# Second Order Action for Gravitational Waves from Inflation

Advanced Cosmology, Mathematical Tripos Part III

Easter Term, 2017

## 1 Statement of the problem and set-ups

**Problem.** The action governing gravitational waves is the second order expansion of the full action

$$S = S_{\mathrm{EH}} + S_\phi \tag{1}$$

where the Einstein–Hilbert action is

$$S_{\mathrm{EH}} = \frac{M_{\mathrm{Pl}}^2}{2} \int \mathrm{d}^4 x \ \sqrt{-g} R \tag{2}$$

and the matter action is

$$S_\phi = \int \mathrm{d}^4 x \ \sqrt{-g} \mathcal{L}_\phi \tag{3}$$

with the scalar-field Lagrangian

$$\mathcal{L}_\phi = -\frac{1}{2} \partial_\mu \phi \partial^\mu \phi - V(\phi). \tag{4}$$

**Set-up.** The perturbed flat FLRW metric $g_{\mu\nu}$ is conformally equivalent to the perturbed Minkowski metric

$$g_{\mu\nu} = a(\eta)^2 \bar{g}_{\mu\nu} = a(\eta)^2 \big(\eta_{\mu\nu} + h_{\mu\nu}\big) \tag{5}$$

where the pure tensor perturbation $h_{\mu\nu}$ is spatial $h_{0\mu} \equiv 0$, traceless $\eta^{\mu\nu} h_{\mu\nu} = 0$ and transverse $\bar{\partial}_\mu h^{\mu\nu} = 0$.

**Convention.** We set $M_{\mathrm{Pl}} = 1$ for now and $(-,+,+,+)$ signature is used. Pre-superscript denotes the order of the quantity. Barred quantities are associated with the perturbed Minkowski metric, and unbarred associated with the perturbed FLRW metric, e.g. $\bar{\partial}^0 = -\partial_\eta$ but $\partial^0 = -a^{-2} \partial_\eta$.

## 2 Preliminary results

In deriving these results, we bear in mind that $h_{\mu\nu}$ is symmetric, purely-spatial, traceless and transverse; its indices are raised with the unperturbed Minkowski metric.

**Result 1.** The perturbed inverse Minkowski metric is

$$\bar{g}^{ab} = \eta^{ab} - h^{ab} + O\big(h^2\big), \tag{6}$$

and the perturbed inverse FLRW metric is $g^{ab} = a^{-2} \bar{g}^{ab}$.

**Result 2.** For conformally equivalent metrices $\tilde{g}_{ab} = \Omega^2 g_{ab}$, the associated Ricci scalars are related by

$$\tilde{R} = \Omega^{-2} \Big( R - 6 \nabla^2 \ln \Omega - 6 \nabla_a \ln \Omega \nabla^a \ln \Omega \Big). \tag{7}$$

*Proof.* Since $\tilde{g}^{ab} = \Omega^{-2} g^{ab}$, direct computation gives

$$\begin{aligned}
\tilde{\Gamma}^a{}_{bc} &= \frac{1}{2}\tilde{g}^{ad}\left(\partial_b \tilde{g}_{cd} + \partial_c \tilde{g}_{bd} - \partial_d \tilde{g}_{bc}\right) \\
&= \Gamma^a{}_{bc} + \frac{1}{2}\Omega^{-2} g^{ad}\left(\tilde{g}_{cd}\partial_b \Omega + \tilde{g}_{bd}\partial_c \Omega - \tilde{g}_{bc}\Omega\partial_d\right) \\
&= \tilde{\Gamma}^a{}_{bc} + \delta^a_c \partial_b \ln\Omega + \delta^a_b \partial_c \ln\Omega - g_{bc}\nabla^a \ln\Omega.
\end{aligned}$$

Hence

$$\begin{aligned}
\tilde{R}_{ab} &= \partial_c \tilde{\Gamma}^c{}_{ab} + \partial_b \tilde{\Gamma}^c{}_{ac} + \tilde{\Gamma}^c{}_{ab}\tilde{\Gamma}^d{}_{cd} - \tilde{\Gamma}^c{}_{ad}\tilde{\Gamma}^d{}_{bc} \\
&= R_{ab} - \partial_c\left(g_{ab}\nabla^c \ln\Omega\right) - 2\partial_a\partial_b \ln\Omega + 2\partial_a \ln\Omega\partial_b \ln\Omega \\
&\qquad + 2\Gamma^c{}_{ab}\partial_c \ln\Omega - 2g_{ab}\partial_c \ln\Omega\partial^c \ln\Omega \\
&\qquad - g_{ab}\Gamma^d{}_{cd}\nabla^c \ln\Omega + g_{bc}\Gamma^c{}_{ad}\nabla^d \ln\Omega + g_{ac}\Gamma^c{}_{bd}\nabla^d \ln\Omega.
\end{aligned}$$

But

$$\begin{aligned}
\partial_c\left(g_{ab}\nabla^c \ln\Omega\right) &= \nabla_c\left(g_{ab}\nabla^c \ln\Omega\right) + g_{bd}\Gamma^d{}_{ac}\nabla^c \ln\Omega + g_{ad}\Gamma^d{}_{bc}\nabla^c \ln\Omega - g_{ab}\Gamma^c{}_{cd}\nabla^d \ln\Omega \\
\partial_a\partial_b \ln\Omega &= \nabla_a\nabla_b \ln\Omega + \Gamma^c{}_{ab}\partial^c \ln\Omega
\end{aligned}$$

so we have

$$\tilde{R}_{ab} = R_{ab} - 2\nabla_a\nabla_b \ln\Omega + 2\nabla_a \ln\Omega\nabla_b \ln\Omega - 2g_{ab}\nabla_c \ln\Omega\nabla^c \ln\Omega - g_{ab}\nabla^2 \ln\Omega.$$

Therefore

$$\begin{aligned}
\tilde{R} &= \Omega^{-2} g^{ab}\tilde{R}_{ab} \\
&= \Omega^{-2}\left(R - 6\nabla^2 \ln\Omega - 6\nabla_a \ln\Omega\nabla^a \ln\Omega\right).
\end{aligned}$$

□

**Result 3.** Using the formula

$$\det(X + \epsilon A) = \det X \det(I + \epsilon B) = \det X\left(1 + \epsilon \operatorname{tr} B + \frac{\epsilon^2}{2}\left[(\operatorname{tr} B)^2 - \operatorname{tr}\left(B^2\right)\right]\right) + O\left(\epsilon^3\right), \quad B \equiv X^{-1}A,$$

we have the perturbed FLRW metric determinant up to second order

$$\begin{aligned}
{}^{(0)}g &= -a^8, \\
{}^{(1)}g &= {}^{(0)}g \operatorname{tr}\left({}^{(0)}g^{\mu\nu}\, a^2 h_{\nu\rho}\right) = {}^{(0)}g \operatorname{tr}\left(a^{-2}\eta^{\mu\nu} a^2 h_{\nu\rho}\right) = 0, \\
{}^{(2)}g &= \frac{1}{2}\,{}^{(0)}g\left[0^2 - \operatorname{tr}\left({}^{(0)}g^{\mu\nu}\, a^2 h_{\nu\rho}\ {}^{(0)}g^{\rho\sigma}\, a^2 h_{\sigma\lambda}\right)\right] = \frac{a^8}{2} h_{\mu\nu}h^{\mu\nu}.
\end{aligned}$$

Thus using binomial expansion $\sqrt{-(g+\delta g)} = \sqrt{-g}\sqrt{1 + g^{-1}\delta g} = \sqrt{-g}\left[1 + \tfrac{1}{2}g^{-1}\delta g - \tfrac{1}{8}g^{-2}\delta g^2 + O\left(\delta g^3\right)\right]$,

$$\begin{aligned}
{}^{(0)}\sqrt{-g} &= a^4, && (8) \\
{}^{(1)}\sqrt{-g} &= \frac{1}{2}\ {}^{(0)}\sqrt{-g}\ \ {}^{(0)}g^{-1}\ \ {}^{(1)}g = 0, && (9) \\
{}^{(2)}\sqrt{-g} &= \frac{1}{2}\ {}^{(0)}\sqrt{-g}\ \ {}^{(0)}g^{-1}\ \ {}^{(2)}g = -\frac{a^4}{4} h_{\mu\nu}h^{\mu\nu}. && (10)
\end{aligned}$$

**Result 4.** The perturbed Minkowski metric Christoffel symbols up to second order are

$$\begin{aligned}
\bar{\Gamma}^0{}_{ij} &= \frac{1}{2}\dot{h}_{ij}, \\
\bar{\Gamma}^i{}_{j0} &= \frac{1}{2}\left(\dot{h}^i{}_j - h^{ik}\dot{h}_{kj}\right) + O\left(h^3\right), \qquad (11) \\
\bar{\Gamma}^i{}_{jk} &= \frac{1}{2}\left(\bar{\partial}_j h^i{}_k + \bar{\partial}_k h^i{}_j - \bar{\partial}^i h_{jk} - h^{il}\bar{\partial}_j h_{lk} - h^{il}\bar{\partial}_k h_{lj} + h^{il}\bar{\partial}_l h_{jk}\right) + O\left(h^3\right)
\end{aligned}$$

and all others are identically zero to all orders.

**Result 5.** We compute the following quantities up to second order

$$\begin{aligned}
\bar{g}^{ab}\bar{\partial}_c\bar{\Gamma}^c{}_{ab} &= -\frac{1}{2}h^{ij}\ddot{h}_{ij} + \frac{1}{2}h^{jk}\bar{\partial}^i\bar{\partial}_i h_{jk},\\
-\bar{g}^{ab}\bar{\partial}_b\bar{\Gamma}^c{}_{ac} &= -\frac{1}{2}\partial_\eta\left(h^{ij}\partial_\eta h_{ij}\right) + \frac{1}{2}\ddot{h}_{ij} + \frac{1}{2}\bar{\partial}^k\left(h^{ij}\bar{\partial}_k h_{ij}\right),\\
\bar{g}^{ab}\bar{\Gamma}^c{}_{ab}\bar{\Gamma}^d{}_{cd} &= 0,\\
-\bar{g}^{ab}\bar{\Gamma}^d{}_{ac}\bar{\Gamma}^c{}_{bd} &= -\frac{1}{4}\dot{h}_{ij}\dot{h}^{ij} + \frac{1}{4}\bar{\partial}_i h_{jk}\bar{\partial}^i h^{jk}
\end{aligned}$$

which add up to give the perturbed Minkowski metric Ricci scalar up to second order

$$\bar{R} = -h^{ij}\ddot{h}_{ij} - \frac{1}{4}\bar{\partial}_i h_{jk}\bar{\partial}^i h^{jk} - \frac{3}{4}\dot{h}_{ij}\dot{h}^{ij} \tag{12}$$

**Result 6.** Combining Result 2 using the conformal factor $\Omega = a$, Result 4 and Result 5, we extract the perturbed FLRW metric Ricci scalar up to second order

$$\begin{aligned}
{}^{(0)}R &= -\frac{6}{a^2}\left(-\partial_\eta^2 \ln a - a^{-2}\dot{a}^2\right) = \frac{6}{a^2}(\mathcal{H}' + \mathcal{H}), && (13)\\
{}^{(1)}R &= 0, && (14)\\
{}^{(2)}R &= -a^{-2}\left(h^{ij}\ddot{h}_{ij} + \frac{1}{4}\bar{\partial}_i h_{jk}\bar{\partial}^i h^{jk} + \frac{3}{4}\dot{h}_{ij}\dot{h}^{ij}\right) - 6a^{-2}\,{}^{(2)}\bar{\Gamma}^i{}_{i0}\,\bar{\partial}^0 \ln a\\
&= -a^{-2}\left(h^{ij}\ddot{h}_{ij} + \frac{1}{4}\bar{\partial}_i h_{jk}\bar{\partial}^i h^{jk} + \frac{3}{4}\dot{h}_{ij}\dot{h}^{ij}\right) - 3a^{-2}\mathcal{H}h^{ij}\dot{h}_{ij}. && (15)
\end{aligned}$$

## 3 Full calculations

### Einstein–Hilbert action

We need to calculate the second order quantity which by (9) is just

$${}^{(2)}\left(\sqrt{-g}R\right) = {}^{(0)}\sqrt{-g}\;{}^{(2)}R + {}^{(2)}\sqrt{-g}\;{}^{(0)}R\,.$$

By (8) and (15)

$${}^{(0)}\sqrt{-g}\;{}^{(2)}R = -a^2\left(h^{ij}\ddot{h}_{ij} + \frac{1}{4}\bar{\partial}_i h_{jk}\bar{\partial}^i h^{jk} + \frac{3}{4}\dot{h}_{ij}\dot{h}^{ij}\right) - 3a^2\mathcal{H}h^{ij}\dot{h}_{ij}.$$

Integrating the first term by parts,

$$-\int \mathrm{d}^4x\, a^2 h^{ij}\ddot{h}_{ij} = \int \mathrm{d}^4x\, 2a^2\mathcal{H}h^{ij}\dot{h}_{ij} + \int \mathrm{d}^4x\, a^2\dot{h}^{ij}\dot{h}_{ij},$$

we find the integral

$$\int \mathrm{d}^4x\;{}^{(0)}\sqrt{-g}\;{}^{(2)}R = \frac{1}{4}\int \mathrm{d}^4x\, a^2\dot{h}^{ij}\dot{h}_{ij} - \frac{1}{4}\int \mathrm{d}^4x\, a^2\bar{\partial}_i h_{jk}\bar{\partial}^i h^{jk} - \int \mathrm{d}^4x\, a^2\mathcal{H}h^{ij}\dot{h}_{ij}.$$

Integrate the last term by parts

$$\begin{aligned}
-\int \mathrm{d}^4x\, a^2\mathcal{H}h^{ij}\dot{h}_{ij} &= \frac{1}{2}\int \mathrm{d}^4x\,\partial_\eta\left(a^2\mathcal{H}\right)h^{ij}h_{ij}\\
&= \frac{1}{2}\int \mathrm{d}^4x\, a^2\left(\dot{\mathcal{H}} + 2\mathcal{H}^2\right).
\end{aligned}$$

Next we have from (10) and (13)

$${}^{(2)}\sqrt{-g}\;{}^{(0)}R = -\frac{3}{2}\int \mathrm{d}^4x\, a^2(\dot{\mathcal{H}} + \mathcal{H}^2)h_{\mu\nu}h^{\mu\nu}.$$

We now arrive at

$$\boxed{{}^{(2)}S_{\text{EH}} = \frac{1}{8}\int \mathrm{d}^4x\, a^2 \dot{h}^{ij}\dot{h}_{ij} - \frac{1}{8}\int \mathrm{d}^4x\, a^2 \bar{\partial}_i h_{jk}\bar{\partial}^i h^{jk} - \frac{1}{4}\int \mathrm{d}^4x\, a^2\left(\mathcal{H}^2 + 2\dot{\mathcal{H}}\right)h^{\rho\sigma}h_{\rho\sigma}.} \tag{16}$$

### Matter action

We need to calculate the second order quantity which by (9) is just

$$^{(2)}(\sqrt{-g}\mathcal{L}_\phi) = {}^{(2)}\sqrt{-g}\ {}^{(0)}\mathcal{L}_\phi + {}^{(0)}\sqrt{-g}\ {}^{(2)}\mathcal{L}_\phi\,.$$

Since $\phi \equiv \phi(t)$,

$$^{(2)}\mathcal{L}_\phi \equiv {}^{(2)}\left[-\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi)\right] = -\frac{1}{2}\,{}^{(2)}g^{\mu\nu}\,\partial_\mu\phi\partial_\nu\phi = -\frac{a^{-4}}{2}h^{ik}h^j{}_k\partial_i\phi\partial_j\phi = 0$$

and we have from (10)

$$^{(2)}(\sqrt{-g}\mathcal{L}_\phi) = {}^{(2)}\sqrt{-g}\ {}^{(0)}\mathcal{L}_\phi = -\frac{a^4}{4}h_{\mu\nu}h^{\mu\nu}\left[\frac{1}{2}a^{-2}\dot{\phi}^2 - V(\phi)\right].$$

But by the Friedman equations written in terms of conformal time

$$\begin{aligned} 3\mathcal{H}^2 &= a^2\rho, \\ -6\dot{\mathcal{H}} &= a^2(\rho + 3P) \end{aligned}$$

where $\rho = \dot{\phi}^2/(2a^2) + V$ and $P = \dot{\phi}^2/(2a^2) - V$, we have

$$^{(2)}(\sqrt{-g}\mathcal{L}_\phi) = {}^{(2)}\sqrt{-g}\ {}^{(0)}\mathcal{L}_\phi = -\frac{a^4}{4}h_{\mu\nu}h^{\mu\nu}P = \frac{a^2}{4}h_{\mu\nu}h^{\mu\nu}\left(\mathcal{H}^2 + 2\dot{\mathcal{H}}\right).$$

Therefore we arrive at

$$\boxed{{}^{(2)}S_\phi = \frac{1}{4}\int \mathrm{d}^4x\, a^2\left(\mathcal{H}^2 + 2\dot{\mathcal{H}}\right)h_{\rho\sigma}h^{\rho\sigma}.} \tag{17}$$

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Finally, adding the two subsectors (16), (17) together and restoring $M_{\text{Pl}}$, we obtain

$$\boxed{{}^{(2)}S = \frac{M_{\text{Pl}}^2}{8}\int \mathrm{d}\eta\, \mathrm{d}^3x\, a^2\left(\dot{h}_{ij}\dot{h}^{ij} - \bar{\partial}_i h_{jk}\bar{\partial}^i h^{jk}\right).} \tag{18}$$

## 4 Extension

In general, the energy-momentum tensor for a single scalar field is

$$T^{\mu\nu} = \partial^\mu\phi\partial^\nu\phi + g^{\mu\nu}\mathcal{L}_\phi \tag{19}$$

so $P \equiv T^i{}_i/3 = {}^{(0)}\mathcal{L}_\phi$ as we saw above. Equivalently the matter action (3) can be recast as

$$\mathcal{L}_\phi = \int \mathrm{d}^4x\ \sqrt{-g}P(X, \phi) \tag{20}$$

where pressure $P$ is a function of both the inflaton field and the kinetic term $X \equiv -g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi/2$. Proceeding as above with the substitution of the Friedman equations yields the same second order action for gravitational waves in FLRW background spacetime, but now valid for general single-field inflation minimally coupled to gravity where the kinetic terms include only first derivatives.